Tarea 6 de particulas

Javier A. Zúñiga C - Sergio Gutiérrez-

October 2016

14)

a) Se define A como $A :=: A : +B$, de ahí que si tenemos:

$$A := a_{j1}a^\dagger_{j2}a_{j3}a^\dagger_{j4}a^\dagger_{j5}$$

Utilizando las relaciones de conmutación:

$$\left[a_k, a^\dagger_l\right] = \delta_{k,l}; \quad [a_k, a_l] = \left[a^\dagger_k, a^\dagger_l\right] = 0$$

Si definimos $\delta_1 = \delta_{j1,j2}$; $\delta_2 = \delta_{j3,j4}$; $\delta_3 = \delta_{j1,j4}$; $\delta_4 = \delta_{j3,j5}$; $\delta_5 = \delta_{j1,j5}$ obtenemos que:

$$\begin{aligned}
a_{j1}a^\dagger_{j2}a_{j3}a^\dagger_{j4}a^\dagger_{j5} &= a^\dagger_{j2}a_{j1}a_{j3}a^\dagger_{j4}a^\dagger_{j5} + \delta_1 a_{j3}a^\dagger_{j4}a^\dagger_{j5} \\
&= a^\dagger_{j2}a_{j1}a^\dagger_{j4}a_{j3}a^\dagger_{j5} + \delta_1 a_{j3}a^\dagger_{j4}a^\dagger_{j5} + a^\dagger_{j2}a_{j1}\delta_2 a^\dagger_{j5} \\
&= a^\dagger_{j2}a^\dagger_{j4}a_{j1}a_{j3}a^\dagger_{j5} + \delta_1 a_{j3}a^\dagger_{j4}a^\dagger_{j5} + a^\dagger_{j2}a_{j1}\delta_2 a^\dagger_{j5} + a^\dagger_{j2}\delta_3 a_{j3}a^\dagger_{j5} \\
&= a^\dagger_{j2}a^\dagger_{j4}a_{j1}a^\dagger_{j5}a_{j3} + \delta_1 a_{j3}a^\dagger_{j4}a^\dagger_{j5} + a^\dagger_{j2}a_{j1}\delta_2 a^\dagger_{j5} + a^\dagger_{j2}\delta_3 a_{j3}a^\dagger_{j5} + a^\dagger_{j2}a^\dagger_{j4}a_{j1}\delta_4 \\
&= a^\dagger_{j2}a^\dagger_{j4}a^\dagger_{j5}a_{j1}a_{j3} + \delta_1 a_{j3}a^\dagger_{j4}a^\dagger_{j5} + a^\dagger_{j2}a_{j1}\delta_2 a^\dagger_{j5} + a^\dagger_{j2}\delta_3 a_{j3}a^\dagger_{j5} + a^\dagger_{j2}a^\dagger_{j4}a_{j1}\delta_4 + a^\dagger_{j2}a^\dagger_{j4}\delta_5 a_{j3}
\end{aligned}$$

necesitamos reorganizar los términos, entonces obtenemos:

$$\begin{aligned}
&= a^\dagger_{j2}a^\dagger_{j4}a^\dagger_{j5}a_{j1}a_{j3} + \delta_1 a^\dagger_{j4}a_{j3}a^\dagger_{j5} + \delta_1\delta_2 a^\dagger_{j5} + \delta_2 a^\dagger_{j2}a^\dagger_{j5}a_{j1} + \delta_2\delta_5 a^\dagger_{j2} \\
&\quad + \delta_3 a^\dagger_{j2}a^\dagger_{j5}a_{j3} + \delta_3\delta_5 a^\dagger_{j2} + \delta_4 a^\dagger_{j2}a^\dagger_{j4}a_{j1} + \delta_5 a^\dagger_{j2}a^\dagger_{j4}a_{j3} \\
&= a^\dagger_{j2}a^\dagger_{j4}a^\dagger_{j5}a_{j1}a_{j3} + \delta_1 a^\dagger_{j4}a^\dagger_{j5}a_{j3} + \delta_1\delta_5 a^\dagger_{j4} + \delta_1\delta_2 a^\dagger_{j5} + \delta_2 a^\dagger_{j2}a^\dagger_{j5}a_{j1} \\
&\quad + \delta_2\delta_5 a^\dagger_{j2} + \delta_3 a^\dagger_{j2}a^\dagger_{j5}a_{j3} + \delta_3\delta_5 a^\dagger_{j2} + \delta_4 a^\dagger_{j2}a^\dagger_{j4}a_{j1} + \delta_5 a^\dagger_{j2}a^\dagger_{j4}a_{j3} \\
&= a^\dagger_{j2}a^\dagger_{j4}a^\dagger_{j5}a_{j1}a_{j3} + \delta_1 a^\dagger_{j4}a^\dagger_{j5}a_{j3} + \delta_2 a^\dagger_{j2}a^\dagger_{j5}a_{j1} + \delta_3 a^\dagger_{j2}a^\dagger_{j5}a_{j3} \\
&\quad + \delta_4 a^\dagger_{j2}a^\dagger_{j4}a_{j1} + \delta_5 a^\dagger_{j2}a^\dagger_{j4}a_{j3} + \delta_1\delta_5 a^\dagger_{j4} + \delta_1\delta_2 a^\dagger_{j5} + \delta_2\delta_5 a^\dagger_{j2} + \delta_3\delta_5 a^\dagger_{j2}
\end{aligned}$$

b) Al aplicar el terorema de Wick, encontramos que:

$$\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\mathcal{O}_5 =: \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\mathcal{O}_5 : + \sum_{i=1}^{4}\sum_{j>i}^{5} \langle 0|\, \mathcal{O}_i\mathcal{O}_j \,|0\rangle : \prod_{k=1\ k\neq j\neq i}^{5} \mathcal{O}_k :$$
$$+ \sum_{i=1}^{2}\sum_{j>i}^{5}\sum_{m>i}^{4}\sum_{n>m}^{5} \langle 0|\, \mathcal{O}_i\mathcal{O}_j \,|0\rangle \, \langle 0|\, \mathcal{O}_m\mathcal{O}_n \,|0\rangle \, \mathcal{O}_p \qquad i \neq j \neq m \neq n \neq p$$

Nótese que el último término podemos hacerle aun una expansión, de tal forma que el resultado será:

$$\begin{aligned}
a_{j1}a^\dagger_{j2}a_{j3}a^\dagger_{j4}a^\dagger_{j5} &= a^\dagger_{j2}a^\dagger_{j4}a^\dagger_{j5}a_{j1}a_{j3} + \langle 0|\, a_{j1}a^\dagger_{j2} \,|0\rangle \, a^\dagger_{j4}a^\dagger_{j5}a_{j3} \\
&+ \langle 0|\, a_{j1}a_{j3} \,|0\rangle \, a^\dagger_{j2}a^\dagger_{j4}a^\dagger_{j5} + \langle 0|\, a_{j1}a^\dagger_{j4} \,|0\rangle \, a^\dagger_{j2}a^\dagger_{j5}a_{j3} + \langle 0|\, a_{j1}a^\dagger_{j5} \,|0\rangle \, a^\dagger_{j2}a^\dagger_{j4}a_{j3} \\
&+ \langle 0|\, a^\dagger_{j2}a_{j3} \,|0\rangle \, a^\dagger_{j4}a^\dagger_{j5}a_{j1} + \langle 0|\, a^\dagger_{j2}a^\dagger_{j4} \,|0\rangle \, a^\dagger_{j5}a_{j1}a_{j3} + \langle 0|\, a^\dagger_{j2}a^\dagger_{j5} \,|0\rangle \, a^\dagger_{j4}a_{j1}a_{j3} \\
&+ \langle 0|\, a_{j3}a^\dagger_{j4} \,|0\rangle \, a^\dagger_{j2}a^\dagger_{j5}a_{j1} + \langle 0|\, a_{j3}a^\dagger_{j5} \,|0\rangle \, a^\dagger_{j2}a^\dagger_{j4}a_{j1} + \langle 0|\, a^\dagger_{j4}a^\dagger_{j5} \,|0\rangle \, a^\dagger_{j2}a_{j1}a_{j3} \\
&+ \langle 0|\, a_{j1}a^\dagger_{j2} \,|0\rangle \, \langle 0|\, a_{j3}a^\dagger_{j4} \,|0\rangle \, a^\dagger_{j5} + \langle 0|\, a_{j1}a^\dagger_{j2} \,|0\rangle \, \langle 0|\, a_{j3}a^\dagger_{j5} \,|0\rangle \, a^\dagger_{j4} + \langle 0|\, a_{j1}a^\dagger_{j2} \,|0\rangle \, \langle 0|\, a^\dagger_{j4}a^\dagger_{j5} \,|0\rangle \, a_{j3} \\
&+ \langle 0|\, a_{j1}a_{j3} \,|0\rangle \, \langle 0|\, a^\dagger_{j2}a^\dagger_{j4} \,|0\rangle \, a^\dagger_{j5} + \langle 0|\, a_{j1}a_{j3} \,|0\rangle \, \langle 0|\, a^\dagger_{j2}a^\dagger_{j5} \,|0\rangle \, a^\dagger_{j4} + \langle 0|\, a_{j1}a_{j3} \,|0\rangle \, \langle 0|\, a^\dagger_{j4}a^\dagger_{j5} \,|0\rangle \, a^\dagger_{j2} \\
&+ \langle 0|\, a_{j1}a^\dagger_{j4} \,|0\rangle \, \langle 0|\, a^\dagger_{j2}a_{j3} \,|0\rangle \, a^\dagger_{j5} + \langle 0|\, a_{j1}a^\dagger_{j4} \,|0\rangle \, \langle 0|\, a^\dagger_{j2}a^\dagger_{j5} \,|0\rangle \, a_{j3} + \langle 0|\, a_{j1}a^\dagger_{j4} \,|0\rangle \, \langle 0|\, a_{j3}a^\dagger_{j5} \,|0\rangle \, a^\dagger_{j2} \\
&+ \langle 0|\, a_{j1}a^\dagger_{j5} \,|0\rangle \, \langle 0|\, a^\dagger_{j2}a_{j3} \,|0\rangle \, a^\dagger_{j4} + \langle 0|\, a_{j1}a^\dagger_{j5} \,|0\rangle \, \langle 0|\, a^\dagger_{j2}a^\dagger_{j4} \,|0\rangle \, a_{j3} + \langle 0|\, a_{j1}a^\dagger_{j5} \,|0\rangle \, \langle 0|\, a_{j3}a^\dagger_{j4} \,|0\rangle \, a^\dagger_{j2} \\
&+ \langle 0|\, a^\dagger_{j2}a_{j3} \,|0\rangle \, \langle 0|\, a^\dagger_{j4}a^\dagger_{j5} \,|0\rangle \, a_{j1} + \langle 0|\, a^\dagger_{j2}a^\dagger_{j4} \,|0\rangle \, \langle 0|\, a_{j3}a^\dagger_{j5} \,|0\rangle \, a_{j1} + \langle 0|\, a^\dagger_{j2}a^\dagger_{j5} \,|0\rangle \, \langle 0|\, a_{j3}a^\dagger_{j4} \,|0\rangle \, a_{j1}
\end{aligned}$$

Ahora, aplicamos los conmutadores y obtenemos que al aplicar un operador destrucción sobre el estado vacío, es igual a 0 por tanto la expresión anterior se puede sumplificar de la siguiente manera:

$$\begin{aligned}
a_{j1}a^\dagger_{j2}a_{j3}a^\dagger_{j4}a^\dagger_{j5} &= a^\dagger_{j2}a^\dagger_{j4}a^\dagger_{j5}a_{j1}a_{j3} + \langle 0|\, a_{j1}a^\dagger_{j2} \,|0\rangle \, a^\dagger_{j4}a^\dagger_{j5}a_{j3} \\
&+ \langle 0|\, a_{j1}a^\dagger_{j4} \,|0\rangle \, a^\dagger_{j2}a^\dagger_{j5}a_{j3} + \langle 0|\, a_{j1}a^\dagger_{j5} \,|0\rangle \, a^\dagger_{j2}a^\dagger_{j4}a_{j3} + \langle 0|\, a^\dagger_{j2}a_{j3} \,|0\rangle \, a^\dagger_{j4}a^\dagger_{j5}a_{j1} \\
&+ \langle 0|\, a_{j3}a^\dagger_{j4} \,|0\rangle \, a^\dagger_{j2}a^\dagger_{j5}a_{j1} + \langle 0|\, a_{j3}a^\dagger_{j5} \,|0\rangle \, a^\dagger_{j2}a^\dagger_{j4}a_{j1} + \langle 0|\, a_{j1}a^\dagger_{j2} \,|0\rangle \, \langle 0|\, a_{j3}a^\dagger_{j4} \,|0\rangle \, a^\dagger_{j5} \\
&+ \langle 0|\, a_{j1}a^\dagger_{j2} \,|0\rangle \, \langle 0|\, a_{j3}a^\dagger_{j5} \,|0\rangle \, a^\dagger_{j4} + \langle 0|\, a_{j1}a^\dagger_{j4} \,|0\rangle \, \langle 0|\, a^\dagger_{j2}a_{j3} \,|0\rangle \, a^\dagger_{j5} + \langle 0|\, a_{j1}a^\dagger_{j4} \,|0\rangle \, \langle 0|\, a_{j3}a^\dagger_{j5} \,|0\rangle \, a^\dagger_{j2} \\
&+ \langle 0|\, a_{j1}a^\dagger_{j5} \,|0\rangle \, \langle 0|\, a^\dagger_{j2}a_{j3} \,|0\rangle \, a^\dagger_{j4} + \langle 0|\, a_{j1}a^\dagger_{j5} \,|0\rangle \, \langle 0|\, a_{j3}a^\dagger_{j4} \,|0\rangle \, a^\dagger_{j2}
\end{aligned}$$

Nuevamente aplicamos los conmutadores de tal manera que los operadores destrucción queden a la derecha:

$$a_{j1}a^\dagger_{j2}a_{j3}a^\dagger_{j4}a^\dagger_{j5} = a^\dagger_{j2}a^\dagger_{j4}a^\dagger_{j5}a_{j1}a_{j3} + \langle 0|\,\delta_1 + a^\dagger_{j2}a_{j1}\,|0\rangle\, a^\dagger_{j4}a^\dagger_{j5}a_{j3}$$
$$+ \langle 0|\,\delta_3 + a^\dagger_{j4}a_{j1}\,|0\rangle\, a^\dagger_{j2}a^\dagger_{j5}a_{j3} + \langle 0|\,\delta_5 + a^\dagger_{j5}a_{j1}\,|0\rangle\, a^\dagger_{j2}a^\dagger_{j4}a_{j3} + \langle 0|\,a^\dagger_{j2}a_{j3}\,|0\rangle\, a^\dagger_{j4}a^\dagger_{j5}a_{j1}$$
$$+ \langle 0|\,\delta_2 + a^\dagger_{j4}a_{j3}\,|0\rangle\, a^\dagger_{j2}a^\dagger_{j5}a_{j1} + \langle 0|\,\delta_4 + a^\dagger_{j5}a_{j3}\,|0\rangle\, a^\dagger_{j2}a^\dagger_{j4}a_{j1} + \langle 0|\,a_{j1}a^\dagger_{j2}\,|0\rangle\,\langle 0|\,a_{j3}a^\dagger_{j4}\,|0\rangle\, a^\dagger_{j5}$$
$$+ \langle 0|\,a_{j1}a^\dagger_{j2}\,|0\rangle\,\langle 0|\,a_{j3}a^\dagger_{j5}\,|0\rangle\, a^\dagger_{j4} + \langle 0|\,a_{j1}a^\dagger_{j4}\,|0\rangle\,\langle 0|\,a^\dagger_{j2}a_{j3}\,|0\rangle\, a^\dagger_{j5} + \langle 0|\,a_{j1}a^\dagger_{j4}\,|0\rangle\,\langle 0|\,a_{j3}a^\dagger_{j5}\,|0\rangle\, a^\dagger_{j2}$$
$$+ \langle 0|\,a_{j1}a^\dagger_{j5}\,|0\rangle\,\langle 0|\,a^\dagger_{j2}a_{j3}\,|0\rangle\, a^\dagger_{j4} + \langle 0|\,a_{j1}a^\dagger_{j5}\,|0\rangle\,\langle 0|\,a_{j3}a^\dagger_{j4}\,|0\rangle\, a^\dagger_{j2}$$

Hay que recordar que un operador destrucción aplicado al vacío es 0, entonces seguismos simplificando de la siguiente forma:

$$a_{j1}a^\dagger_{j2}a_{j3}a^\dagger_{j4}a^\dagger_{j5} = a^\dagger_{j2}a^\dagger_{j4}a^\dagger_{j5}a_{j1}a_{j3} + \langle 0|\,\delta_1\,|0\rangle\, a^\dagger_{j4}a^\dagger_{j5}a_{j3}$$
$$+ \langle 0|\,\delta_3\,|0\rangle\, a^\dagger_{j2}a^\dagger_{j5}a_{j3} + \langle 0|\,\delta_5\,|0\rangle\, a^\dagger_{j2}a^\dagger_{j4}a_{j3}$$
$$+ \langle 0|\,\delta_2\,|0\rangle\, a^\dagger_{j2}a^\dagger_{j5}a_{j1} + \langle 0|\,\delta_4\,|0\rangle\, a^\dagger_{j2}a^\dagger_{j4}a_{j1}$$
$$+ \langle 0|\,\delta_1\,|0\rangle\,\langle 0|\,\delta_2\,|0\rangle\, a^\dagger_{j5} + \langle 0|\,\delta_1\,|0\rangle\,\langle 0|\,\delta_5\,|0\rangle\, a^\dagger_{j4}$$
$$+ \langle 0|\,\delta_2\,|0\rangle\,\langle 0|\,\delta_5\,|0\rangle\, a^\dagger_{j2} + \langle 0|\,\delta_3\,|0\rangle\,\langle 0|\,\delta_5\,|0\rangle\, a^\dagger_{j2}$$
$$= a^\dagger_{j2}a^\dagger_{j4}a^\dagger_{j5}a_{j1}a_{j3} + \delta_1 a^\dagger_{j4}a^\dagger_{j5}a_{j3} + \delta_2 a^\dagger_{j2}a^\dagger_{j5}a_{j1} + \delta_3 a^\dagger_{j2}a^\dagger_{j5}a_{j3}$$
$$+\delta_4 a^\dagger_{j2}a^\dagger_{j4}a_{j1} + \delta_5 a^\dagger_{j2}a^\dagger_{j4}a_{j3} + \delta_1\delta_5 a^\dagger_{j4} + \delta_1\delta_2 a^\dagger_{j5} + \delta_2\delta_5 a^\dagger_{j2} + \delta_3\delta_5 a^\dagger_{j2}$$

Nótese que obtuvimos los mismos términos que en el literal anterior

c)

$$: a_k a^\dagger_l + a_k a_l :: a_i a^\dagger_j :$$
$$=: (a^\dagger_l a_k + a_k a_l)(a^\dagger_j a_i) :$$
$$=: a^\dagger_l a_k a^\dagger_j a_i + a_k a_l a^\dagger_j a_i :$$
$$= a^\dagger_l a^\dagger_j a_k a_i + \delta_{j,k} a^\dagger_l a_i + a^\dagger_j a_k a_l a_i + \delta_{j,k} a_l a_i + \delta_{j,l} a_k a_i 0$$

Consideremos ahora $: (a_k a^\dagger_l + a_k a_l)(a_i a^\dagger_j) :$ y comparemos con los resultados obtenidos anteriormente:

$$: (a_k a^\dagger_l + a_k a_l)(a_i a^\dagger_j) :$$
$$=: (a^\dagger_l a_k + \delta_{l,k} + a_k a_l)(a^\dagger_j a_i + \delta_{i,j}) :$$
$$=: a^\dagger_l a_k a^\dagger_j a_i + a_k a_l a^\dagger_j a_i + \delta_{l,k} a^\dagger_j a_i + a^\dagger_l a_k \delta_{i,j} + \delta_{l,k}\delta_{i,j} + a_k a_l \delta_{i,j} :$$
$$= a^\dagger_l a^\dagger_j a_k a_i + \delta_{j,k} a^\dagger_l a_i + a^\dagger_j a_k a_l a_i + \delta_{j,k} a_l a_i + \delta_{j,l} a_k a_i$$
$$+\delta_{l,k} a^\dagger_j a_i + a^\dagger_l a_k \delta_{i,j} + \delta_{l,k}\delta_{i,j} + a_k a_l \delta_{i,j}$$
$$=: a_k a^\dagger_l + a_k a_l :: a_i a^\dagger_j : +\delta_{l,k} a^\dagger_j a_i + a^\dagger_l a_k \delta_{i,j} + \delta_{l,k}\delta_{i,j} + a_k a_l \delta_{i,j}$$

Como era de esperarse el segundo resultado es igual al primero más unos términos que corresponden a las contracciones interiores de los operadores. De esta manera se verificó la propiedad.

15)

a) Tenemos el campo $\Phi(x)$ definido de en términos de operadores escalera así que en primera instancia vamos a ver cual es el resultado al conmutar dos campos.

$$\Phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2} E_p \left(a(p)e^{-ipx} + a^\dagger(p)e^{ipx} \right) \ , [\Phi(x), \Phi(y)] = \Phi(x)\Phi(y) - \Phi(y)\Phi(x)$$

$$[\Phi(x), \Phi(y)] = \int \left(\frac{d^3p}{2E_p} \right) \left(\frac{d^3q}{2E_q} \right) \left[\left(a(p)e^{-ipx} + a^\dagger(p)e^{ipx} \right), \left(a(q)e^{-iqy} + a^\dagger(q)e^{iqy} \right) \right]$$

$$[\text{""}, \text{""}] = a(p)a(q)e^{-ipx}e^{-iqy} + a(p)a^\dagger(q)e^{-ipx}e^{iqy} + a^\dagger(p)a(q)e^{ipx}e^{-iqy} + a^\dagger(p)a^\dagger(q)e^{ipx}e^{iqy}$$
$$-a(q)a(p)e^{-ipx}e^{-iqy} - a(q)a^\dagger(p)e^{ipx}e^{-iqy} - a^\dagger(q)a(p)e^{-ipx}e^{iqy} - a^\dagger(q)a^\dagger(p)e^{ipx}e^{iqy}$$
$$= (a(p)a^\dagger(q) - a^\dagger(q)a(p))e^{-ipx}e^{iqy} + (a^\dagger(p)a(q) - a(q)a^\dagger(p))e^{ipx}e^{-iqy}$$
$$= [a(p), a^\dagger(q)]e^{iqy-ipx} + [a^\dagger(p), a(q)])e^{ipx-iqy}$$

En la ultima excreción encontramos que el conmutador del integrando se puede reescribir en términos de conmutadores de operadores construcción los cuales cumplen con las reglas de conmutación $[a(p), a^\dagger(q)] = 2E_p\delta(p-q), [a(p), a(q)] = [a^\dagger(p), a^\dagger(q)] = 0$ así que si lo remplazamos en las integrales tenemos que solo se va a tener una integral cuando $p = q$ así que el conmutador entre dos campos es:

$$[\Phi(x), \Phi(y)] = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E_p} \left(-e^{ipy-ipx} - e^{ipy-ipx} \right)$$
$$= \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E_p} \left(e^{-ip(x-y)} - e^{ip(x-y)} \right)$$
$$\Phi(x)\Phi(y) - \Phi(y)\Phi(x) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E_p} \left(e^{-ip(x-y)} - e^{ip(x-y)} \right) \big|_{p^0=E_p}$$

Con lo anterior podemos encontrar que la forma explicita de el prducto entre dos campos: $\Phi(x)\Phi(y) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E_p} e^{-ip(x-y)}$, $\Phi(y)\Phi(x) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E_p} e^{ip(x-y)}$ así con esto ya definido podemos encontrar el producto temporalmente ordenado:

$$T\Phi(x)\Phi(y) := \Theta(x^0 - y^0)\Phi(x)\Phi(y) + \Theta(y^0 - x^0)\Phi(y)\Phi(x)$$

ahora sabemos:

$$\langle 0| \Phi(x)\Phi(y) |0\rangle = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E_p} e^{-ip(x-y)} \ , \ \Theta(u) = \frac{1}{2\pi i} \int d\frac{e^{i\lambda u}}{\lambda - i\epsilon}$$

Asi que para calcular $\langle 0|\,\Phi(x)\Phi(y)\Theta(x^0-y^0)\,|0\rangle$ podemos multiplicar las dos integrales y mirar que podemos concluir el producto.

$$\langle 0|\,\Phi(x)\Phi(y)\Theta(x^0-y^0)\,|0\rangle = \frac{1}{i(2\pi)^4}\int d\lambda \frac{e^{i\lambda(x^0-y^0)}}{\lambda - i\epsilon}\int \frac{d^3p}{2E_p}e^{-ip(x-y)}$$

Ahora hagamos un cambio de variable donde $k^0 := E_p - \lambda\ ,\ k := p$ así podemos hacer el cambio de dos integrales a una sola sobre un cuatro-momento.

$$\frac{1}{i(2\pi)^4}\int d\lambda \frac{e^{i\lambda(x^0-y^0)}}{\lambda - i\epsilon}\int \frac{d^3p}{2E_p}e^{-ip(x-y)} = \frac{-i}{(2\pi)^4}\int \frac{d^4k}{2E_p}\frac{e^{-ik(x-y)}}{E_p - k^0 - i\epsilon}$$

Ahora para el segundo termino del producto temporalmente ordenado, tiene el mismo calculo que el que hemos calculado acá solo con unas discrepancias en los símbolos.

$$\Phi(y)\Phi(x) = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2E_p}e^{ip(x-y)}\ ,\ \Theta(-u) = \frac{-1}{2\pi i}\int d\frac{e^{i\lambda u}}{\lambda + i\epsilon}$$

$$\langle 0|\,\Phi(y)\Phi(x)\Theta(-(x^0-y^0))\,|0\rangle = \frac{i}{(2\pi)^4}\int d\lambda \frac{e^{i\lambda(x^0-y^0)}}{\lambda - i\epsilon}\int \frac{d^3p}{2E_p}e^{ip(x-y)}$$

Ahora aplicamos un cambio de variable con el otro signo,$k^0 := -E_p + \lambda\ ,\ k := p$ entonces el calculo es análogo:

$$\langle 0|\,\Phi(y)\Phi(x)\Theta(-(x^0-y^0))\,|0\rangle = \frac{-i}{(2\pi)^4}\int \frac{d^4k}{2E_p}\frac{e^{-ik(x-y)}}{E_p + k^0 - i\epsilon}$$

Ahora cuando juntemos las integrales tenemos que reducir los denominadores.

$$\frac{1}{E_p - k^0 - i\epsilon} + \frac{1}{E_p + k^0 - i\epsilon} = \frac{2E_k}{E_k^2 - (k^0)^2 - 2iE_k\epsilon}$$
$$= \frac{2E_k}{m^2 + k^2 - (k^0)^2 + 2iE_k\epsilon} = \frac{2E_k}{m^2 + k^2 - i\epsilon}$$

Así que si reagrupamos todos lo terminos tenemos que:

$$\langle 0|\,T\Phi(x)\Phi(y)\,|0\rangle = \frac{i}{(2\pi)^4}\int \frac{e^{-k(x-y)}}{k^2 - m^2 + i\epsilon}$$

b) Si consideramos la interacción entre 4 campos : $\Phi(x)^4$: que se puede considerar como dos partículas q_1, q_2 y después salen dos partículas p_1, p_2 ahora como cada una de estas partículas es descrita por un campo $\Phi = (a + a^\dagger)$. Ahora solo van a sobre vivir los terminos donde tengamos la misma cantidad de operadores construcion que destrucion asi que son 6 terminos en total en numero de premutacion entre los elemntos daga asi que en forma integral tenemos:

$$S^{(1)} = -i\lambda\int d^4x : \Phi(x)^4 := -i6\lambda\int d^4x f^*_{k1}f^*_{k2}f_{k3}f_{k4}a^\dagger_{k1}a^\dagger_{k1}a_{k3}a_{k4} = -i\lambda\int d^4x : \Phi(x)^4$$
$$= \frac{-i6\lambda}{(2\pi)^6}\int (d^4x)\left(\frac{d^3k1}{2E_{k1}}\right)\left(\frac{d^3k2}{2E_{k2}}\right)\left(\frac{d^3k3}{2E_{k3}}\right)\left(\frac{d^3k4}{2E_{k4}}\right)e^{i(k1+k2-k3-k4)x}a^\dagger_{k1}a^\dagger_{k1}a_{k3}a_{k4}$$

si recordamos la definición de delta lo aplicamos al caso anterior,así exigiendo la conservación del momento

$$S^{(1)} = \frac{-i6\lambda(2\pi)^4}{(2\pi)^6}\int\left(\frac{d^3k1}{2E_{k1}}\right)\left(\frac{d^3k2}{2E_{k2}}\right)\left(\frac{d^3k3}{2E_{k3}}\right)\left(\frac{d^3k4}{2E_{k4}}\right)\delta(k1+k2-k3-k4)a^\dagger_{k1}a^\dagger_{k1}a_{k3}a_{k4}$$

ahora apliquemos y calculemos los elementos matriciales $\langle in|\, S^{(1)}\, |out\rangle$

$$|in\rangle = a^\dagger_{pa}a^\dagger_{pb}\,|0\rangle \;\;,\;\; |in\rangle = a^\dagger_{qc}a^\dagger_{qd}\,|0\rangle$$

ahora por las reglas de comnutacion y por las contracion de los operadores podesmo concluir como en la primera parte del problema que podemos arreglar la intgral al darnos cuenta que cualquier contracion de los elementos es ko mismo asi que tenemos cuatro contracciones posibles asi que:

$$\begin{aligned}\langle in|\, S^{(1)}\, |out\rangle &= \langle 0|\, a^\dagger_{pa}a^\dagger_{pb}S^{(1)}a^\dagger_{qc}a^\dagger_{qd}\,|0\rangle \\ &= \frac{-i(6)(4)\lambda}{(2\pi)^2}\delta(k1+k2-k3-k4) \\ \langle in|\, S^{(1)}\, |out\rangle &= -i\lambda\frac{6}{\pi^2}\delta(k1+k2-k3-k4)\end{aligned}$$